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MOMENT DISTRIBUTIONS OF PHASE TYPE

MOGENS BLADT AND BO FRIIS NIELSEN

ABSTRACT. In this paper we prove that the class of distributions on the positive reals with a rational Laplace transform, also known as matrix-exponential distributions, is closed under formation of moment distributions. In particular, the results are hence valid for the well known class of phase-type distributions. We construct representations for moment distributions based on a general matrix-exponential distribution which turns out to be a generalization of the moment distributions based on exponential distributions. For moment distributions based on phase-type distributions we find an appropriate alternative representation in terms of sub-intensity matrices. Finally we are able to find explicit expressions for both the Lorenz curve and the Gini index.

1. INTRODUCTION

The class of phase-type distributions is well established in the area of applied probability, see [6]. In particular in insurance risk and queueing theory it has become a standard method for establishing exact results of complex models, either in terms of explicit or algorithmic solutions. Most models with underlying phase-type distributions can be solved entirely by probabilistic arguments. The class of phase-type distributions is contained in the broader class of matrix-exponential distributions, which are the distributions on the positive reals which have a rational Laplace transform. Probabilistic arguments are not directly applicable for this entire class. Any distribution on the positive reals may be approximated arbitrarily close (in terms of weak convergence) by phase-type and hence also by matrix-exponential distributions.

Both matrix-exponential and phase-type distributions have a number of important closure properties. Among those are the distributions of the age and residual life-time of a stationary renewal process with inter-arrivals of either type. In this paper we show that the spread, which is the sum of the age and residual life-time, is also phase-type distributed. Moreover, we give some explicit representations. The spread is known to have a first order moment distribution, and we prove that moment distributions of any order are again matrix-exponentially or phase-type distributed. Other distributions which are closed under the formation of moment distributions encompasses distributions such as log-normal, Pareto and gamma.

For the first order distribution we present an explicit formula for the related Lorenz curve and Gini index. Moment distributions of orders one, two and three have been extensively used in areas such as economy, physics, demography and civil engineering.

The paper is organized as follows. In section 2 we provide the necessary background on moment distributions. Section 3 contains basic definitions and results related to

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matrix-exponential and phase-type distributions. In section 4 we prove our main results and in section 5 we calculate the Lorenz curve and Gini index.

2. MOMENT DISTRIBUTIONS

Consider the density function f of a non-negative random variable X . Then the functions

$$f_i(x) = \frac{x^i f(x)}{\mu_i}, \quad \text{where} \quad \mu_i = \int_0^\infty x^i f(x) dx,$$

are again densities of some non-negative random variables $X^{(i)}$ provided that μ_i exists. We say that f_i is the i 'th order moment distribution of f . Moment distributions are also known under the names length-biased distributions ($i = 1$) and size biased distributions ($i > 1$). The explanation for the names of the latter terms will be apparent in a moment.

We let M denote the operator that for a given density f returns its corresponding first order moment distribution. Hence, if $g = M(f)$ then $g(x) = xf(x)/\mu_1$. Then it is easy to see that

$$(1) \quad f_{i+1} = M(f_i),$$

which means that the $(i + 1)$ 'th order moment distribution of f is also the first order moment distribution of f_i . In particular, $\mathbb{E}(X^{(i)}) = \mu_{i+1}/\mu_i$.

Consider a stationary renewal process with inter-arrival time distribution given by the density f . This is a delayed renewal process where the time until the first arrival has density $f_e(x) = (1 - F(x))/\mu_1 = \bar{F}(x)/\mu_1$, where F is the distribution function corresponding to f and \bar{F} denotes the survival function.

Let F_e denote the distribution function corresponding to f_e . Let A_t denote the age of the process at time t , i.e. the time from last arrival to t and R_t the residual life-time, which is the time from t until the next arrival. The joint distribution of A_t and R_t is given by (see e.g. [8], Proposition 3.2).

$$(2) \quad \mathbb{P}(A_t > x, R_t > y) = \bar{F}_e(x + y).$$

Differentiating (2) with respect to x and y , we obtain that the joint density of A_t and R_t is given by

$$f_{(A_t, R_t)}(x, y) = \frac{f(x + y)}{\mu_1}.$$

The spread $S_t = A_t + R_t$ is the length of the inter-arrival interval which contains the time point t and is then calculated to have density

$$f_{S_t}(x) = \int_0^x f_{(A_t, R_t)}(x - t, t) dt = \frac{xf(x)}{\mu_1}.$$

The distribution of S_t is stochastically larger than the usual inter-arrival distribution f , which is due to the fact that at any time t , it is more likely that a large inter-arrival interval contains t than a smaller. This situation is often called the inspection or waiting time paradox. Hence, the first order moment distribution appears in a natural way as the distribution of the length of the inter-arrival interval in progress at time t . This also explains the alternative name of length biased distribution.

Size biasing in dimensions two and three occur in a similar fashion. Consider a map of a country which is partitioned into counties. We could be interested in estimating the distribution of the county sizes. Sampling counties through realizing a two dimensional

Poisson process over the map results in an area biased distribution, where the probability of choosing a county is proportional to its area. Three dimensional size-biased distributions occurs when estimating grain sizes by using sieves of increasing fineness and then using the weight of the different groups, which is approximately proportional to the cube of the grain diameters, see e.g. [9].

3. MATRIX-EXPONENTIAL AND PHASE-TYPE DISTRIBUTIONS

A distribution of a non-negative random variable is called matrix-exponential if it has a density function f on the form $f(x) = \boldsymbol{\alpha} \exp(\mathbf{S}x)\mathbf{s}$, where $\boldsymbol{\alpha}$ and \mathbf{s} are row and column vectors respectively, \mathbf{S} is a matrix and \exp denotes the matrix-exponential defined in the usual way by series expansion $\exp(\mathbf{S}x) = \sum_{n=0}^{\infty} \mathbf{S}^n x^n / n!$. The triple $(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ is called a representation of the matrix-exponential distribution. The dimension of the vectors and the matrix must of course be compatible. The common dimension p is referred to as the order of the matrix-exponential representation. The order is called minimal if it is not possible to find another representation of f with a strictly lower order in which it is called the order of the distribution.

A distribution f being matrix-exponential is equivalent to its Laplace transform $\hat{f}(s) = \int_0^{\infty} e^{-sx} f(x) dx$ being a rational function, i.e. a fraction between two polynomials. The main results of the paper are valid for matrix-exponential distributions in general, however, we shall additionally specialize the theory to the sub-class of phase-type distributions. Phase-type distributions contain more structure than the matrix-exponential distributions and allow for probabilistic reasoning. Since phase-type distributions are more commonly used in applications than matrix-exponential distributions in general, we shall accommodate the theory to the latter class which basically consists in reorganizing representations of the distributions.

A phase-type distribution is the distribution of the first exit time from a finite set of transient states in a continuous time Markov jump process. Let $\{X_t\}_{t \geq 0}$ be a Markov jump process with transient states $1, 2, \dots, p$. For the purpose of calculating the distribution of the first exit time, we can assume without loss of generality that there is only one additional absorbing state. We may then write the intensity matrix of $\{X_t\}_{t \geq 0}$, $\mathbf{\Lambda}$, in a block partitioned way as

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{S} & \mathbf{s} \\ \mathbf{0} & 0 \end{pmatrix},$$

where \mathbf{S} is a $p \times p$ matrix of transition rates between the transient states, \mathbf{s} is a column vector of rates for jumps to the absorbing states, and $\mathbf{0}$ is a p -dimensional row vector of zeroes. Since the rows of an intensity matrix necessarily sum to 0, we have that $\mathbf{s} = -\mathbf{S}\mathbf{e}$, where \mathbf{e} is the p -dimensional column vector of ones. Let $\boldsymbol{\alpha}$ denote the initial distribution of $\{X_t\}_{t \geq 0}$ concentrated on the transient states, i.e. $\mathbb{P}(X_0 = i) = \alpha_i$, $\alpha_1 + \dots + \alpha_p = 1$. Then it is easy to prove that the density function f of the time until absorption is given by $f(x) = \boldsymbol{\alpha} \exp(\mathbf{S}x)\mathbf{s}$. This is of course a matrix-exponential distribution but with a special structure on the vector $\boldsymbol{\alpha}$ and matrix \mathbf{S} . Since \mathbf{s} is given in terms of \mathbf{S} , for phase-type distributions we only need $(\boldsymbol{\alpha}, \mathbf{S})$ in order to specify the distribution and we refer to the couple as a representation. The n 'th moment of a matrix exponential distribution with representation $(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ is given by $(-1)^{n+1} n! \boldsymbol{\alpha} \mathbf{S}^{-(n+1)} \mathbf{s}$. If $\mathbf{s} = -\mathbf{S}\mathbf{e}$, which is true for phase-type distributions, the formula for the n 'th moment reduces to $n! \boldsymbol{\alpha} (-\mathbf{S})^{-n} \mathbf{e}$. The cumulative distribution functions are respectively given by $1 + \boldsymbol{\alpha} \exp(\mathbf{S}x) \mathbf{S}^{-1} \mathbf{s}$ and $1 - \boldsymbol{\alpha} \exp(\mathbf{S}x) \mathbf{e}$. Finally

it should be mentioned that it is always possible to represent a matrix-exponential distribution $(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ in such a way that $\mathbf{s} = -\mathbf{S}\mathbf{e}$ and $\boldsymbol{\alpha}\mathbf{e} = 1$. For more details on matrix-exponential and phase-type distributions we refer to [2] and [3].

Representations of matrix-exponential and phase-type distributions are rarely unique. If for example $(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ is a representation of a matrix-exponential distribution and \mathbf{M} is a non-singular matrix, then $(\boldsymbol{\alpha}\mathbf{M}^{-1}, \mathbf{M}\mathbf{S}\mathbf{M}^{-1}, \mathbf{M}\mathbf{s})$ is another representation of the same distribution.

Of special interest is the case where $\mathbf{M} = \boldsymbol{\Delta}(\mathbf{m})$ is a diagonal matrix with $\mathbf{m} = (m_1, m_2, \dots, m_p)$ and $m_i \neq 0$ for all i in which case we may construct the time-reversed (or dual) representation by letting $\mathbf{m} = -\boldsymbol{\alpha}\mathbf{S}^{-1}$ for a phase-type representation $(\boldsymbol{\alpha}, \mathbf{S})$. The time reversed representation $(\hat{\boldsymbol{\alpha}}, \hat{\mathbf{S}})$ is then given by $\hat{\boldsymbol{\alpha}} = \mathbf{s}'\mathbf{M}$ and $\hat{\mathbf{S}} = \mathbf{M}^{-1}\mathbf{S}'\mathbf{M}$. For more details, see [1, 4, 7].

Consider a delayed renewal process, where the distribution of the time until the first arrival is matrix-exponentially distributed with representation $(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ and the remaining inter-arrivals are matrix-exponentially distributed with common representation $(\boldsymbol{\beta}, \mathbf{S}, \mathbf{s})$. Then the renewal density is given by $u(x) = \boldsymbol{\alpha} \exp((\mathbf{S} + \mathbf{s}\boldsymbol{\beta})x)\mathbf{s}$ and the residual life-time R_t is matrix-exponentially distributed with representation $(\boldsymbol{\beta} \exp((\mathbf{S} + \mathbf{s}\boldsymbol{\alpha})t), \mathbf{S}, \mathbf{s})$ (see [2] for details and further properties). The renewal process is stationary if the delay is matrix-exponentially distributed with representation $(-\boldsymbol{\alpha}\mathbf{S}^{-1}/\boldsymbol{\alpha}\mathbf{S}^{-2}\mathbf{s}, \mathbf{S}, \mathbf{s})$. For the stationary phase-type renewal process, the delay distribution is of phase-type with representation $(\boldsymbol{\alpha}\mathbf{S}^{-1}/\boldsymbol{\alpha}\mathbf{S}^{-1}\mathbf{e}, \mathbf{S})$.

4. MOMENT DISTRIBUTIONS GENERATED BY MATRIX-EXPONENTIAL DISTRIBUTIONS

Let f be the density of a matrix-exponential distribution with representation $(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$. Then it is readily clear that any moment distribution f_i is again a matrix-exponential distribution. This follows directly from the i 'th derivative of the Laplace transform \hat{f} is given by

$$\frac{d^i}{ds^i} \hat{f}(s) = \int_0^\infty (-1)^i x^i e^{-sx} f(x) dx = (-1)^i \mu_i \hat{f}_i(s).$$

Since $\hat{f}(s)$ is a rational function, so is its i 'th derivative, and hence $\hat{f}_i(s)$ is also a rational function which, in turn, is equivalent to f_i being matrix-exponential. We next address the problem of finding representations for such distributions.

It is often an advantage, in particular in applications of non-linear models, that a matrix-exponential distribution with representation $(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ satisfies $\mathbf{s} = -\mathbf{S}\mathbf{e}$ and $\boldsymbol{\alpha}\mathbf{e} = 1$, see [3]. In the following we shall restrict attention to matrix-exponential distributions satisfying this property.

Theorem 4.1. *Consider a matrix-exponential distribution with representation $(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{s})$ such that $\mathbf{s} = -\mathbf{S}\mathbf{e}$. Then its n 'th moment distribution is again matrix-exponential with representation $(\boldsymbol{\alpha}_n, \mathbf{S}_n, \mathbf{s}_n)$, where*

$$\boldsymbol{\alpha}_n = \left(\frac{\boldsymbol{\alpha}\mathbf{S}^{-n}}{\boldsymbol{\alpha}\mathbf{S}^{-n}\mathbf{e}}, 0, \dots, 0 \right) \quad \mathbf{S}_n = \begin{pmatrix} \mathbf{S} & -\mathbf{S} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{S} & -\mathbf{S} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S} \end{pmatrix}, \quad \mathbf{s}_n = \begin{pmatrix} 0 \\ 0 \\ \dots \\ \mathbf{s} \end{pmatrix}$$

Proof: When calculating $\alpha_n \exp(\mathbf{S}_n x) \mathbf{s}_n$ we see that the only upper right term of importance in the matrix-exponential. This term is calculated to be $((-1)^n/n!)x^n \mathbf{S}^n \exp(\mathbf{S}x)$ by taking powers of $\hat{\mathbf{S}}$ and summing up. Thus

$$\alpha_n e^{\mathbf{S}_n x} \mathbf{s}_n = \frac{\alpha \mathbf{S}^{-n}}{\alpha \mathbf{S}^{-n} \mathbf{e}} \frac{(-1)^n}{n!} x^n \mathbf{S}^n e^{\mathbf{S} x} \mathbf{s} = \frac{x^n \alpha e^{\mathbf{S} x} \mathbf{s}}{(-1)^n n! \alpha \mathbf{S}^{-n} \mathbf{e}} = \frac{x^n \alpha e^{\mathbf{S} x} \mathbf{s}}{\mu_n},$$

where μ_n (as usual) denotes the n 'th moment of the matrix-exponential distribution. \square

A similar calculation yields the following

Corollary 4.2.

$$F_n(x) = 1 - \frac{\alpha \mathbf{S}^{-n}}{\alpha \mathbf{S}^{-n} \mathbf{e}} \sum_{i=0}^n \frac{(-x \mathbf{S})^i}{i!} e^{\mathbf{S} x} \mathbf{e}$$

If the underlying distribution is a phase-type distribution with representation (α, \mathbf{S}) , then the above representation for the n 'th moment distribution will only yield a phase-type representation when it is exponential, since otherwise there will be negative off-diagonal elements present. In the case of phase-type distributions, the above representation will therefore be of limited use.

We still need to prove that the n 'th moment distribution of a phase-type distribution is again of phase-type. To this end it is sufficient to prove that the first order moment distribution is of phase-type since the n 'th order can be obtained by an iterative argument (see underneath).

Theorem 4.3. *Consider a phase-type distribution with representation (α, \mathbf{S}) . Then the first moment distribution is again of phase-type with representation (α_1, \mathbf{S}_1) , where*

$$\alpha_1 = (\mathbf{m} \bullet \mathbf{s}, 0), \quad \mathbf{S}_1 = \begin{pmatrix} \Delta^{-1}(\mathbf{m}) \mathbf{S}' \Delta(\mathbf{m}) & \Delta(\mathbf{m}) \Delta(-\mathbf{m} \mathbf{S}) \\ \mathbf{0} & \mathbf{S} \end{pmatrix}, \quad \mathbf{m} = -\frac{\alpha \mathbf{S}^{-2}}{\alpha \mathbf{S}^{-1} \mathbf{e}}$$

Proof: Consider a stationary renewal process with inter-arrival distribution which is phase-type with representation (α, \mathbf{S}) . Then, at time t , the age A_t and residual life R_t are both distributed according to the stationary distribution F_e (see (2)) and hence phase-type distributed with representation (π_1, \mathbf{S}) , where $\pi_1 = \alpha \mathbf{S}^{-1} / \alpha \mathbf{S}^{-1} \mathbf{e}$. The distribution of the spread, $A_t + B_t$, is the first order moment distribution of the phase-type distribution with representation (α, \mathbf{S}) . The distribution of the spread can be established using a phase-type argument as follows.

Since A_t is phase-type distributed with representation (π_1, \mathbf{S}) it can also be represented in terms of its time-reversed representation $(\hat{\alpha}, \hat{\mathbf{S}})$, where $\hat{\alpha} = \mathbf{s}' \Delta(\mathbf{m})$, $\hat{\mathbf{S}} = \Delta(\mathbf{m})^{-1} \mathbf{S}' \Delta(\mathbf{m})$ and $\hat{\mathbf{s}} = -\hat{\mathbf{S}} \mathbf{e} = \Delta(\mathbf{m}) \pi_1'$, where $\mathbf{m} = -\pi_1 \mathbf{S}^{-1} = -\frac{\alpha \mathbf{S}^{-2}}{\alpha \mathbf{S}^{-1} \mathbf{e}}$. The time reversed distribution exits with distribution π_1 , the same as which the B_t initiates. Thus the spread is phase-type distributed with representation $\alpha_1 = (\hat{\alpha}, \mathbf{0})$ and

$$\mathbf{S}_1 = \begin{pmatrix} \hat{\mathbf{S}} & \Delta(\hat{\mathbf{s}}) \\ \mathbf{0} & \mathbf{S} \end{pmatrix}.$$

Now $\hat{\alpha} = \mathbf{s}' \Delta(\mathbf{m}) = \mathbf{m} \bullet \mathbf{s} = -\pi_1 \mathbf{S}^{-1} \bullet \mathbf{s} = -\frac{\alpha \mathbf{S}^{-2}}{\alpha \mathbf{S}^{-1} \mathbf{e}} \bullet \mathbf{s}$. Using $\mathbf{m} = -\pi_1 \mathbf{S}^{-1} = -\frac{\alpha \mathbf{S}^{-2}}{\alpha \mathbf{S}^{-1} \mathbf{e}}$ we see that $\hat{\mathbf{S}}$ also has the desired form. \square

Since the second order moment distribution f_2 can be obtained as the first order moment distribution of f_1 (see equation (1)), then by applying Theorem 4.3 to the phase-type representation $(\boldsymbol{\alpha}_1, \mathbf{S}_1)$ we see that f_2 again of phase-type. If the order of $(\boldsymbol{\alpha}, \mathbf{S})$ is p , then the order of f_1 is $2p$ and the order of f_2 is $4p$. Proceeding in this way, we see that the n 'th order moment distribution f_n is of phase-type of order $2^n p$. In the following we show an alternative phase-type representation of f_n of order $(n+1)p$.

Theorem 4.4. *Consider a phase-type distribution with representation $(\boldsymbol{\alpha}, \mathbf{S})$. Then the n 'th order moment distribution is again of phase-type with representation $(\boldsymbol{\alpha}_n, \mathbf{S}_n)$, where*

$$\begin{aligned} \boldsymbol{\alpha}_n &= \left(\frac{\rho_{n+1}}{\rho_n} \boldsymbol{\pi}_{n+1} \bullet \mathbf{s}, 0, \dots, 0 \right) \\ \mathbf{S}_n &= \begin{bmatrix} \mathbf{C}_{n+1} & \mathbf{D}_{n+1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_n & \mathbf{D}_n & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{n-1} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{C}_2 & \mathbf{D}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{C}_1 \end{bmatrix} \end{aligned}$$

and $\rho_i = \mu_i / i! = \boldsymbol{\alpha}(-\mathbf{S})^{-i} \mathbf{e}$ are the reduced moments,

$$\boldsymbol{\pi}_i = \rho_i^{-1} \boldsymbol{\alpha}(-\mathbf{S})^{-i}, \quad \mathbf{C}_i = \boldsymbol{\Delta}(\boldsymbol{\pi}_i)^{-1} \mathbf{S}' \boldsymbol{\Delta}(\boldsymbol{\pi}_i) \quad \text{and} \quad \mathbf{D}_i = \frac{\rho_{i-1}}{\rho_i} \boldsymbol{\Delta}(\boldsymbol{\pi}_i)^{-1} \boldsymbol{\Delta}(\boldsymbol{\pi}_{i-1}).$$

Proof: We lack a probabilistic argument and the proof is purely analytical. We prove that the Laplace transform corresponding to $(\boldsymbol{\alpha}_n, \mathbf{S}_n)$ is identical to the one of Theorem 4.1. Noticing that $(u\mathbf{I} - \boldsymbol{\Delta}^{-1} \mathbf{S}' \boldsymbol{\Delta})^{-1} = \boldsymbol{\Delta}^{-1} (u\mathbf{I} - \mathbf{S}')^{-1} \boldsymbol{\Delta}$ it is readily seen that the Laplace transform $L(u)$ corresponding to $(\boldsymbol{\alpha}_n, \mathbf{S}_n)$, which amounts to

$$\begin{aligned} L(u) &= \left(\frac{\rho_{n+1}}{\rho_n} \mathbf{s} \bullet \boldsymbol{\pi}_{n+1} \right) \left[\prod_{i=0}^{n-1} (u\mathbf{I} - \boldsymbol{\Delta}(\boldsymbol{\pi}_{n+1-i})^{-1} \mathbf{S}' \boldsymbol{\Delta}(\boldsymbol{\pi}_{n+1-i}))^{-1} \right. \\ &\quad \left. \boldsymbol{\Delta}(\boldsymbol{\pi}_{n+1-i})^{-1} \boldsymbol{\Delta}(\boldsymbol{\pi}_{n-i}) \frac{\rho_{n-i}}{\rho_{n+1-i}} \right] (u\mathbf{I} - \boldsymbol{\Delta}(\boldsymbol{\pi}_1)^{-1} \mathbf{S}' \boldsymbol{\Delta}_1)^{-1} \boldsymbol{\Delta}(\boldsymbol{\pi}_1)^{-1} \boldsymbol{\pi}'_0 \frac{1}{\rho_1} \\ &= \rho_n^{-1} \mathbf{s}' (u\mathbf{I} - \mathbf{S}')^{-n-1} \boldsymbol{\alpha}' \\ &= \rho_n^{-1} \boldsymbol{\alpha} (u\mathbf{I} - \mathbf{S})^{-(n+1)} \mathbf{s} \\ &= \boldsymbol{\pi}_n [(u\mathbf{I} - \mathbf{S})^{-1} (-\mathbf{S})]^n (u\mathbf{I} - \mathbf{S})^{-1} \mathbf{s} \end{aligned}$$

coinciding with the Laplace transform of Theorem 4.1. \square

Please notice that for $n = 1$, the representation above provides an alternative to the one in Theorem 4.3.

5. LORENZ CURVE AND THE GINI INDEX

If F is a distribution function and F_1 the distribution function of the corresponding first order moment distribution, then the parametric curve $\gamma : t \rightarrow (F(t), F_1(t)), t \in [0, \infty)$ is called the Lorenz curve. Traditionally the Lorenz curve is used to illustrate inequality in a society with the interpretation that the poorest $x = F(t)$ per cent of the population posses $y = F_1(t)$ per cent of the total wealth. Another measure frequently used is the Gini index, which is defined as twice the area between the curve γ and the

line $y = x$. It compares the area between the curve γ and the line $y = x$ to the area of the triangle under the line $y = x$ for $x = [0, 1]$, which is one half. The larger the Gini index, the larger the inequality of incomes. If γ is the straight line $y = x$, then there would be complete equality with the Gini index being zero, while if $\gamma(t) = (F(t), 0)$ for all t and $\lim_{t \rightarrow \infty} \gamma(t) = (1, 1)$, then there would be complete inequality and Gini index would be one.

If F is matrix-exponentially distributed we are able to provide explicit formulas for both the Lorenz curve and the Gini index.

Theorem 5.1. *Let F be the distribution function of a matrix-exponential distribution with representation $(\alpha, \mathbf{S}, \mathbf{s})$, where $\mathbf{s} = -\mathbf{S}\mathbf{e}$. Then the Lorenz curve is given by the formula*

$$\gamma : t \rightarrow \left(1 - \alpha e^{\mathbf{S}t} \mathbf{e}, 1 - \frac{\alpha \mathbf{S}^{-1}}{\alpha \mathbf{S}^{-1} \mathbf{e}} \left(e^{\mathbf{S}t} \mathbf{e} + t e^{\mathbf{S}t} \mathbf{s} \right) \right)$$

and the Gini index G by

$$G = 2(\alpha \otimes \alpha_1) (-(\mathbf{S} \oplus \mathbf{S}_1))^{-1} (\mathbf{s} \otimes \mathbf{e}) - 1.$$

Proof: The Lorenz curve representation follows from Corollary 4.2 with $n = 1$. The area A under the curve γ is given by

$$\begin{aligned} A &= \int_0^\infty F'(t) F_1(t) dt \\ &= \int_0^\infty \alpha e^{\mathbf{S}t} \mathbf{s} \left(1 - \alpha_1 e^{\mathbf{S}_1 t} \mathbf{e} \right) dt \\ &= 1 - \int_0^\infty \alpha e^{\mathbf{S}t} \mathbf{s} \alpha_1 e^{\mathbf{S}_1 t} \mathbf{e} dt \\ &= 1 + (\alpha \otimes \alpha_1) (\mathbf{S} \oplus \mathbf{S}_1)^{-1} (\mathbf{s} \otimes \mathbf{e}), \end{aligned}$$

where the last equality follows from [5] and where \otimes and \oplus are the Kronecker product and sum respectively. The result now follows as $G = 2(\frac{1}{2} - A)$. \square

Example 5.2. Consider the densities

$$\begin{aligned} f(x) &= 4xe^{-2x}, \\ g(x) &= 9e^{-10x} + \frac{1}{91}e^{-10x/91}, \\ h(x) &= \frac{2}{3}e^{-x}(1 + \cos(x)). \end{aligned}$$

The former two are phase-type with representations

$$\left((1, 0), \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix} \right), \quad \left(\left(\frac{9}{10}, \frac{1}{10} \right), \begin{pmatrix} -10 & 0 \\ 0 & -\frac{10}{91} \end{pmatrix} \right)$$

respectively while the latter is a matrix-exponential distribution not being phase-type with representation

$$\left((0, 0, 1), \begin{pmatrix} -1 & 0 & 0 \\ -\frac{2}{3} & -1 & 1 \\ \frac{2}{3} & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{4}{3} \end{pmatrix} \right).$$

The three cases are plotted in Figures 1, 2 and 3.

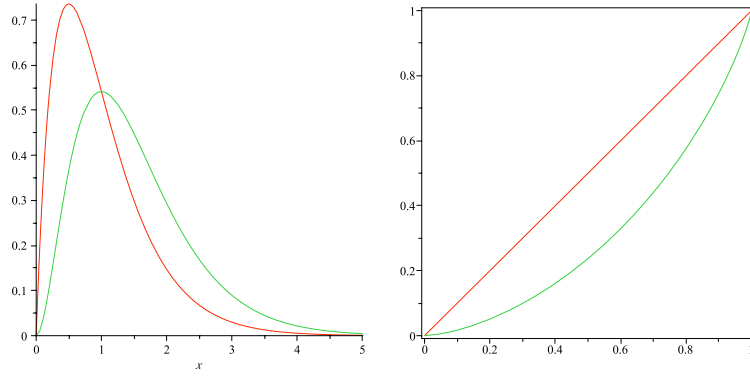


FIGURE 1. Left: Densities f and f_1 . Right: Corresponding Lorenz curve. The Gini index is 0.3750.

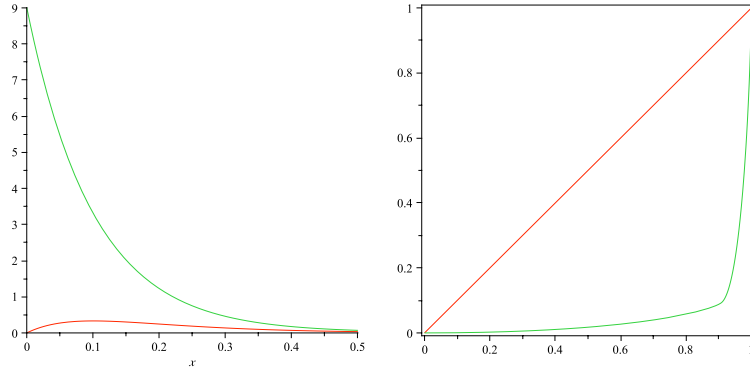


FIGURE 2. Left: Densities g and g_1 . Right: Corresponding Lorenz curve. The Gini index is 0.8962.

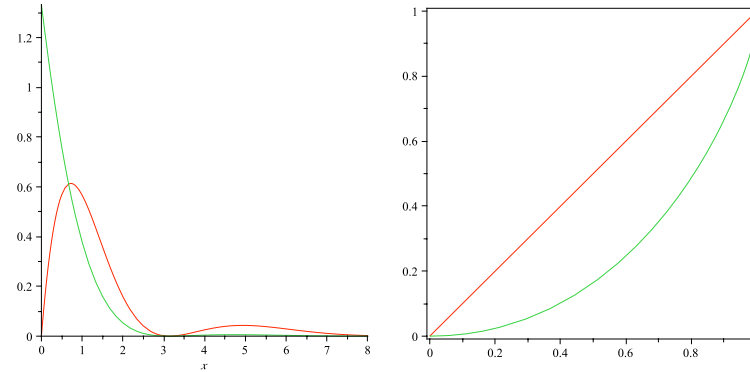


FIGURE 3. Left: Densities h and h_1 . Right: Corresponding Lorenz curve. The Gini index is 0.4917.

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